Exact and efficient simulation of photon quantum interference using tensor train decomposition

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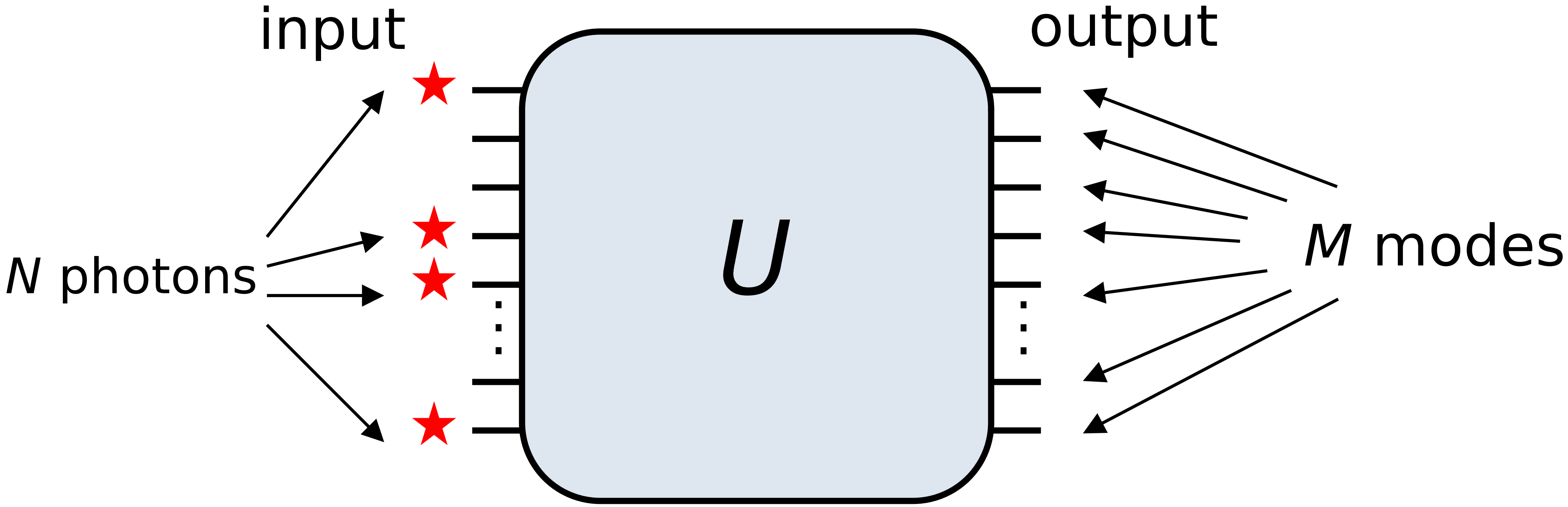
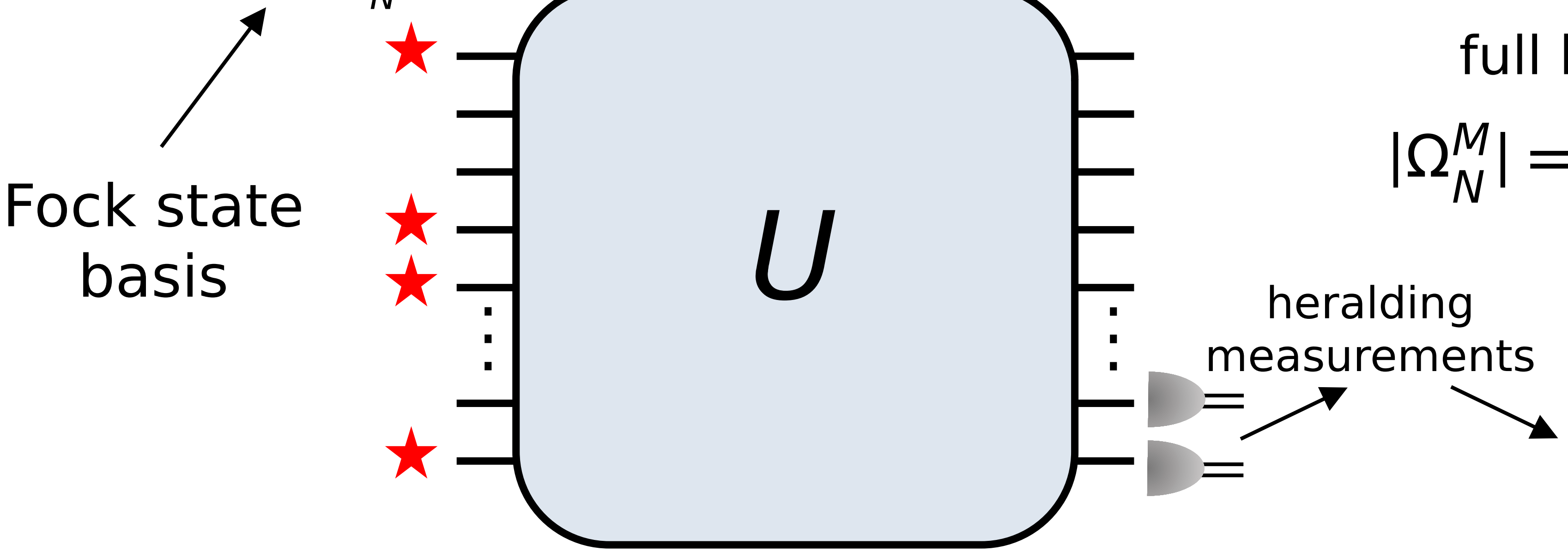
Problem under consideration

A standart setup of linear optical quanutm computation



Problem under consideration

$$|\Psi^{(in)}\rangle = \sum_{\boldsymbol{S}\in\Omega_N^M} C_{\boldsymbol{S}}^{(in)}|\boldsymbol{S}\rangle \longrightarrow |\Psi^{(out)}\rangle = \hat{\mathcal{U}}|\Psi^{(in)}\rangle = \sum_{\boldsymbol{T}\in\Omega_N^M} C_{\boldsymbol{T}}^{(out)}|\boldsymbol{T}\rangle$$



full basis:

$$|\Omega_N^M| = \binom{M-1+N}{N}$$

may be restricted

The task: precisely compute $C_{\boldsymbol{T}}^{(out)}$ for a lot of different U

Fock state

occupation numbers → assignment list: $1 \le s_1 \le s_2 \le \cdots \le s_N \le M$

$$|\boldsymbol{S}\rangle = |S_1, S_2, S_3 \ldots, S_M\rangle = |\boldsymbol{s}\rangle = |\{s_1, s_2, s_3 \ldots, s_N\}\rangle$$

S_i photons in mode i $\qquad \sum_{i=1}^{M} S_i = N$

"i-th photon in mode s_i" $\qquad \sum_{j=1}^{N} \delta_{i s_j} = S_i$

examples:

$$|1,2,0,1\rangle = |\{1, 2, 2, 4\}\rangle$$

$$|0,2,4\rangle = |\{2, 2, 3, 3, 3, 3\}\rangle$$

full basis:

$$|\Omega_N^M| = \binom{M-1+N}{N}$$

Fock operators

creation operator

$$\Omega_N^M \xrightarrow{\hat{a}_i^\dagger} \Omega_{N+1}^M$$

annihilation operator

$$\Omega_N^M \xrightarrow{\hat{a}_i} \Omega_{N-1}^M$$

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

Definition:

$$[\hat{a}_i, \hat{a}_j] = 0, \; [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$$

$$\hat{a}_i^\dagger |\ldots, S_i, \ldots\rangle = \sqrt{S_i + 1} |\ldots, S_i + 1, \ldots\rangle \qquad \hat{a}_i |\ldots, S_i, \ldots\rangle = \sqrt{S_i} |\ldots, S_i - 1, \ldots\rangle$$

$$\hat{a}^\dagger = \begin{pmatrix} 0 & 0 & \ldots & & \\ 1 & 0 & \ldots & & \\ 0 & \sqrt{2} & \ldots & & \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & \ldots & \sqrt{K} & 0 \\ & & \ldots & 0 & \sqrt{K+1} \end{pmatrix} \updownarrow \Omega_{K+1}^1, \quad \overset{\Omega_K^1}{\leftrightarrow}$$

$$\hat{a} = \begin{pmatrix} 0 & 1 & 0 & \ldots & & \\ 0 & 0 & \sqrt{2} & \ldots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & \ldots & \sqrt{K-1} & 0 \\ & & & \ldots & 0 & \sqrt{K} \end{pmatrix} \updownarrow \Omega_{K-1}^1, \quad \overset{\Omega_K^1}{\leftrightarrow}$$

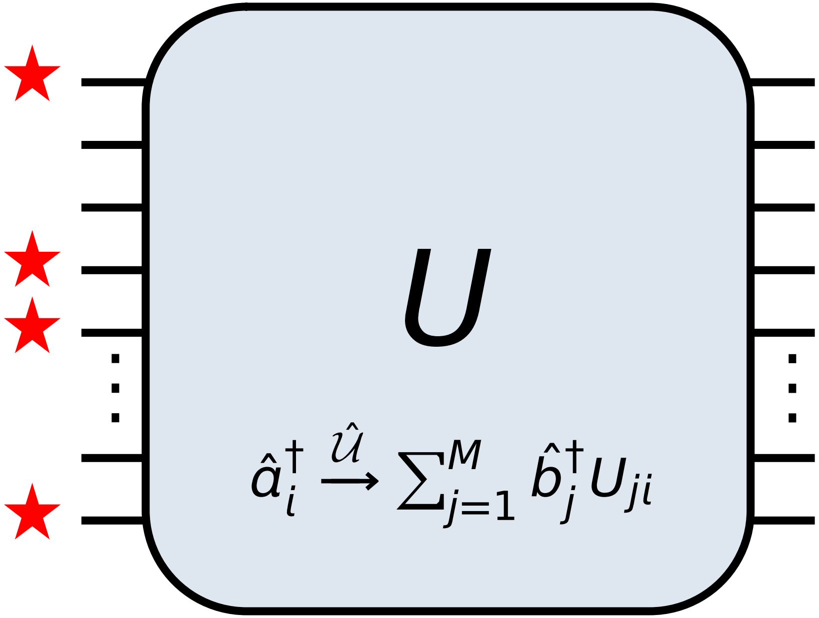
very sparse

Fock state evolution

$$\mathbf{s}! = \prod_{i=1}^{M} S_i!$$

$$|\mathbf{S}\rangle = |S_1, S_2, S_3 \cdots, S_M\rangle = \prod_{i=1}^{M} \frac{(\hat{a}_i^\dagger)^{S_i}}{\sqrt{S_i!}} |vac\rangle = |\mathbf{s}\rangle = |\{s_1, s_2, s_3 \cdots, s_N\}\rangle = \frac{1}{\sqrt{\mathbf{s}!}} \prod_{i=1}^{N} \hat{a}_{s_i}^\dagger |vac\rangle$$

transformation:



input state

$$|\mathbf{s}\rangle = \frac{1}{\sqrt{\mathbf{s}!}} \prod_{i=1}^{N} \hat{a}_{s_i}^\dagger |vac\rangle$$

$$\downarrow \hat{\mathcal{U}}$$

output state

$$|\Psi^{(out)}\rangle = \frac{1}{\sqrt{\mathbf{s}!}} \prod_{i=1}^{N} \left(\sum_{j=1}^{M} \hat{b}_j^\dagger U_{j s_i} \right) |vac\rangle = \sum_{\mathbf{t} \in \Omega_N^M} \mathcal{U}_{\mathbf{ts}} |\mathbf{t}\rangle$$

transformation amplitude

$$\mathcal{U}_{\mathbf{ts}} = \frac{\text{perm}[U^{\mathbf{ts}}]}{\sqrt{\mathbf{s}!\mathbf{t}!}}$$

$O(N2^N)$

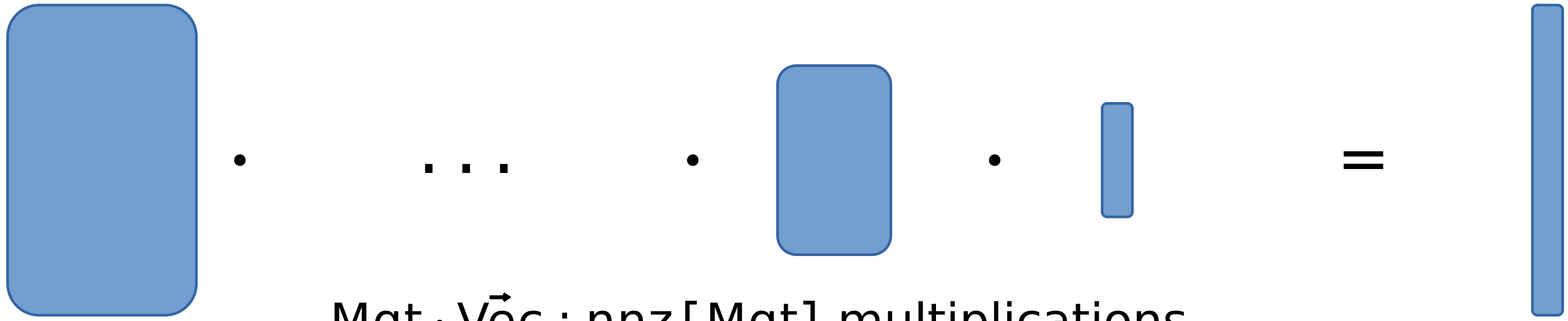
$$[U^{\mathbf{ts}}]_{ij} \equiv U_{t_i s_j}, \ \text{perm}[A] \equiv \sum_{\sigma \in \mathcal{S}_N} \prod_{i=1}^{N} A_{\sigma_i i}$$

Total complexity of a problem is $\sim O\left(N 2^N \binom{M-1+N}{N}\right)$

SLOS method

$$\frac{1}{\sqrt{\boldsymbol{s}!}}\prod_{i=1}^{N}\left(\sum_{j=1}^{M}\hat{b}_j^\dagger U_{js_i}\right)|vac\rangle = \frac{1}{\sqrt{\boldsymbol{s}!}}\left(\sum_{j=1}^{M}\hat{b}_j^\dagger U_{js_1}\right)\dots\left(\sum_{j=1}^{M}\hat{b}_j^\dagger U_{js_{N-1}}\right)\cdot\left(\sum_{j=1}^{M}\hat{b}_j^\dagger U_{js_N}\right)|vac\rangle = |\Psi^{(out)}\rangle$$

$\Omega_N^M \times \Omega_{N-1}^M$ $\quad$ $\Omega_2^M \times \Omega_1^M$ $\quad$ $\Omega_1^M \times \Omega_0^M$ $\quad$ Ω_N^M vector



$\mathrm{M\alpha t} \cdot \vec{\mathrm{Vec}} : \mathrm{nnz}[\mathrm{M\alpha t}]$ multiplications

Matrices have a lot of zeroes — total complexity of a problem is $O\left(N\binom{M-1+N}{N}\right)$

Generalization to arbitrary set of input and output Fock bases is unclear

Nicolas Heurtel, at al. (2023). «Strong simulation of linear optical processes» Computer Physics Communications, 291, 108848.

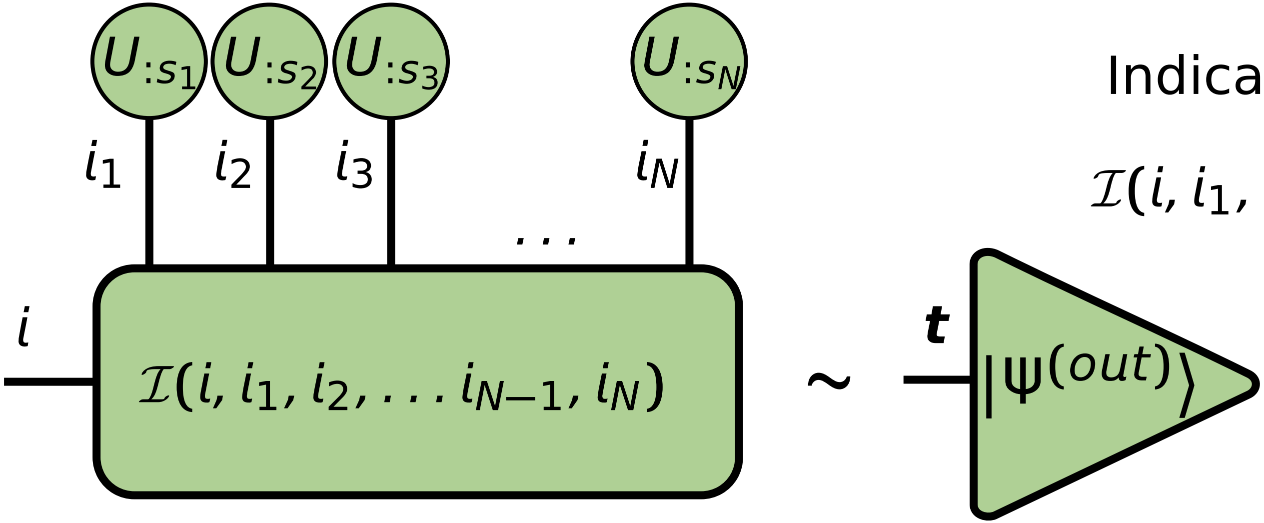
TT decomposition

may not be a full basis

Suppose output basis is given by a set of assignment lists: $L[i] = \{t_1, t_2, \ldots t_N\}$

A key observation:

$$\mathrm{perm}\left[U^{\boldsymbol{ts}}\right] = \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^{N} U_{t_{\sigma_j} s_j} \overset{\boldsymbol{t}=L[i]}{=} \boldsymbol{t}! \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \cdots \sum_{i_N=1}^{M} \mathcal{I}(i, i_1, i_2, \ldots i_N) \prod_{k=1}^{N} U_{i_k s_k}$$



Indicator tensor:

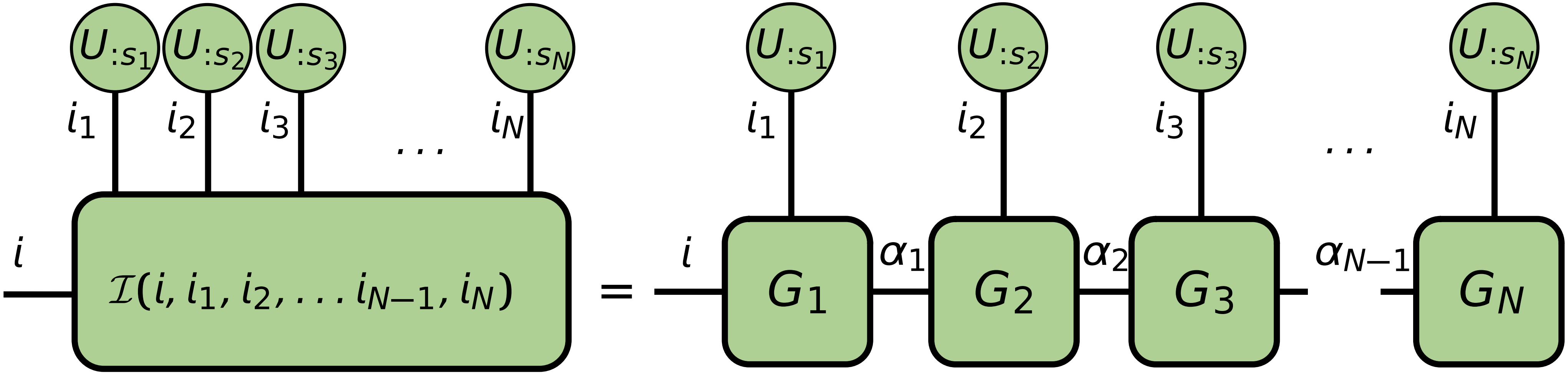
$$\mathcal{I}(i, i_1, \ldots i_N) = \begin{cases} 1, \{i_1, \ldots i_N\} = L[i] \\ 0, \text{otherwise} \end{cases}$$

$\mathcal{I}$ is completely defined by L

TT decomposition

cores of TT

$$\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\cdots\sum_{i_N=1}^{M}\mathcal{I}(i,i_1,i_2,\dots i_N)\prod_{k=1}^{N}U_{i_k s_k} = \sum_{\alpha_1\alpha_2\dots\alpha_{N-1}}\prod_{k=1}^{N}\left(\sum_{i_k=1}^{M}G_k(\alpha_{k-1};i_k;\alpha_k)U_{i_k s_k}\right)$$



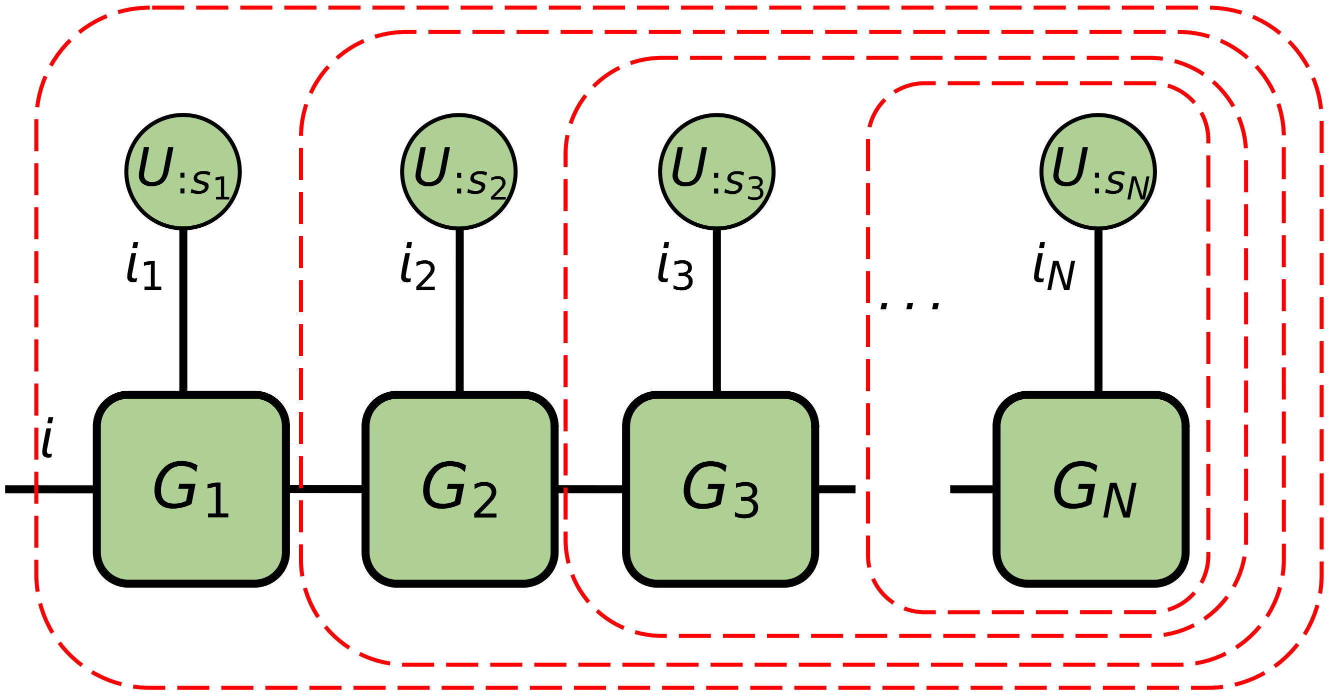
Derivative function decomposition method introduced in

Ryzhakov, G., & Oseledets, I. (2023). «Constructive TT-representation of the tensors given as index interaction functions with applications». The Eeventh Internlational Conference on Learning Representations

Single input, arbitrary output

A universal recipe for single Fock input and arbitrary Fock basis output

order of evaluation



only matrix by vector multiplication

input: $\boldsymbol{s}$

output: $L[i]$

for full output basis:

$$O\left(N\binom{M-1+N}{N}\right)$$

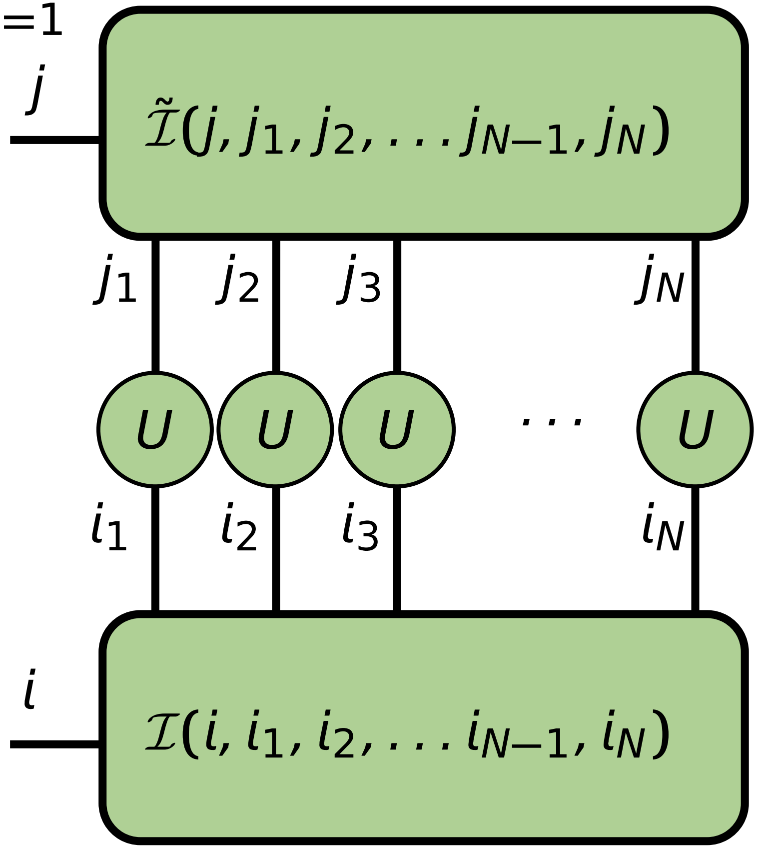
equivalent to SLOS

Arbitrary input, arbitrary output

$$\mathcal{U}_{\mathbf{ts}} \sim \sum_{i_1 j_1=1}^{M} \sum_{i_2 j_2=1}^{M} \cdots \sum_{i_N j_N=1}^{M} \mathcal{I}(i, i_1, i_2, \ldots i_N)\tilde{\mathcal{I}}(j, j_1, j_2, \ldots j_N) \prod_{k=1}^{N} U_{i_k j_k}$$

$J[j] = \{s_1, s_2, \ldots s_N\}$

input set of assignment lists



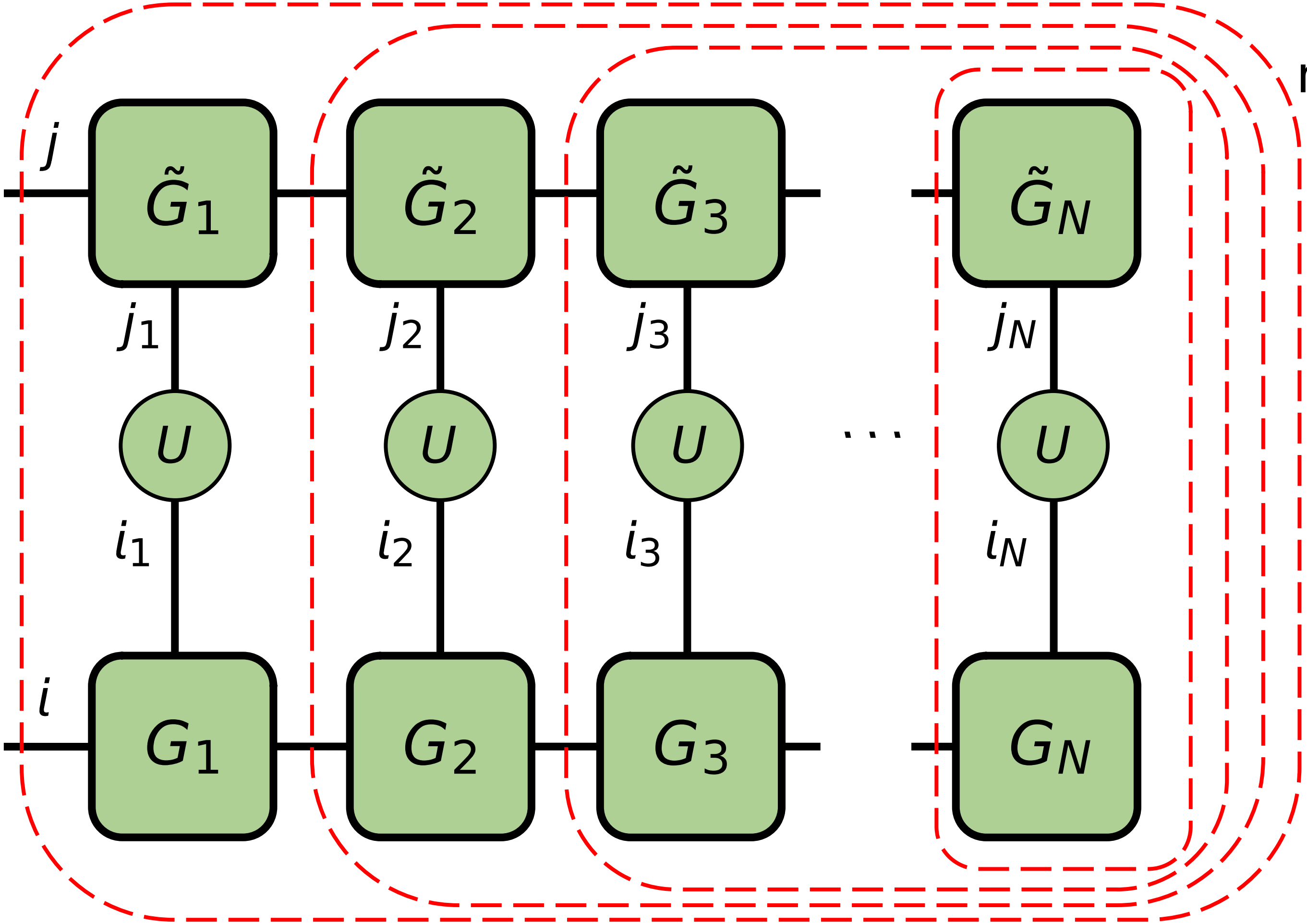
semi-indicator tensor encoding the input Fock basis

output set of assignment lists

indicator tensor encoding the output Fock basis

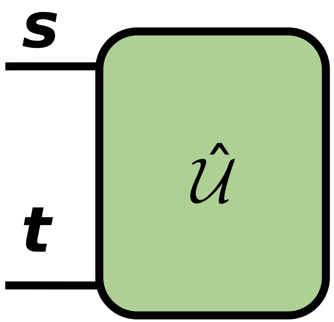
$L[i] = \{t_1, t_2, \ldots t_N\}$

Arbitrary input, arbitrary output



$\sim$

matrix of transformation amplitudes

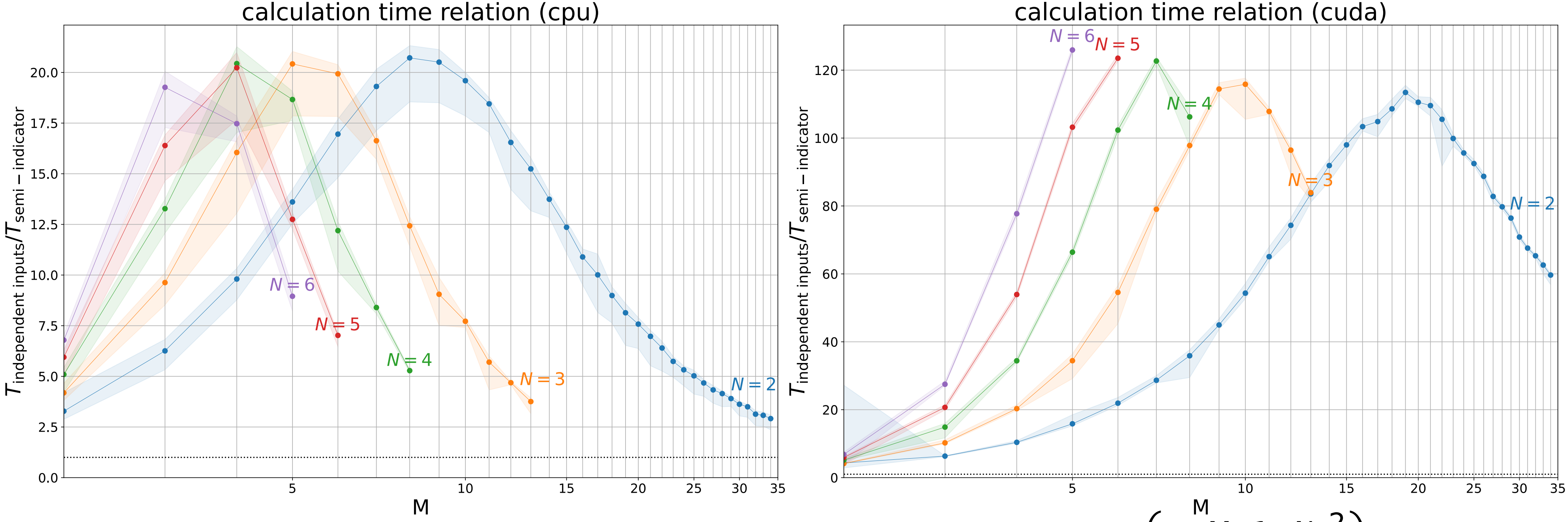


for full output and input basis:

$$O\left(N\binom{M-1+N}{N}^2\right)$$

Numerical comparison

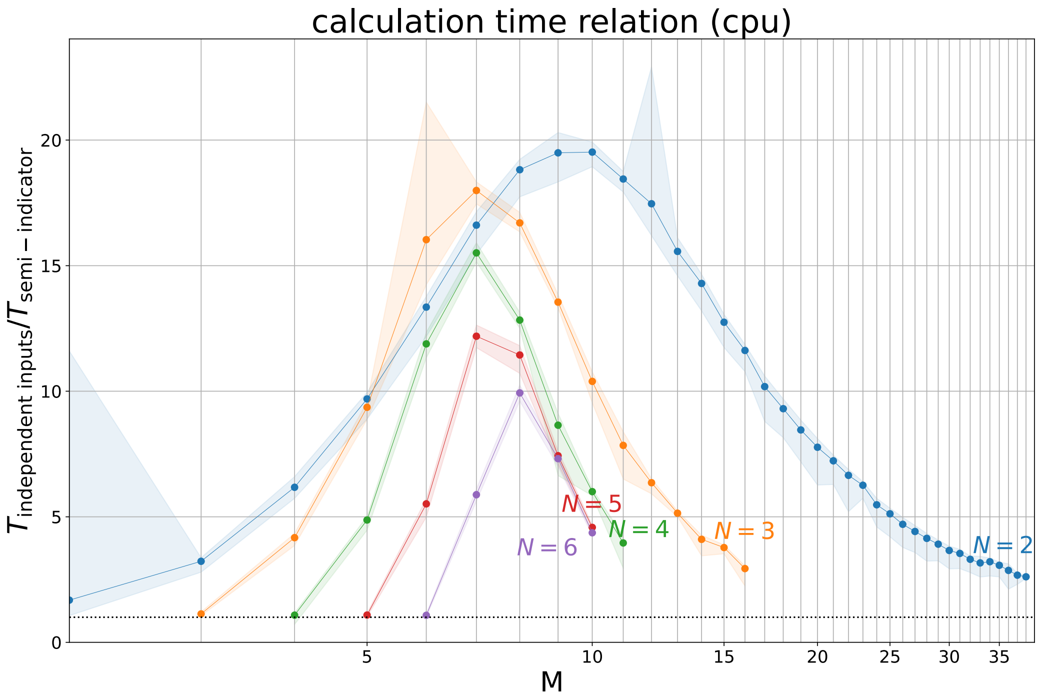
How faster is it to calculate transformation amplitudes with the semi-indicator tensor than to calculate each input separately for full bases?

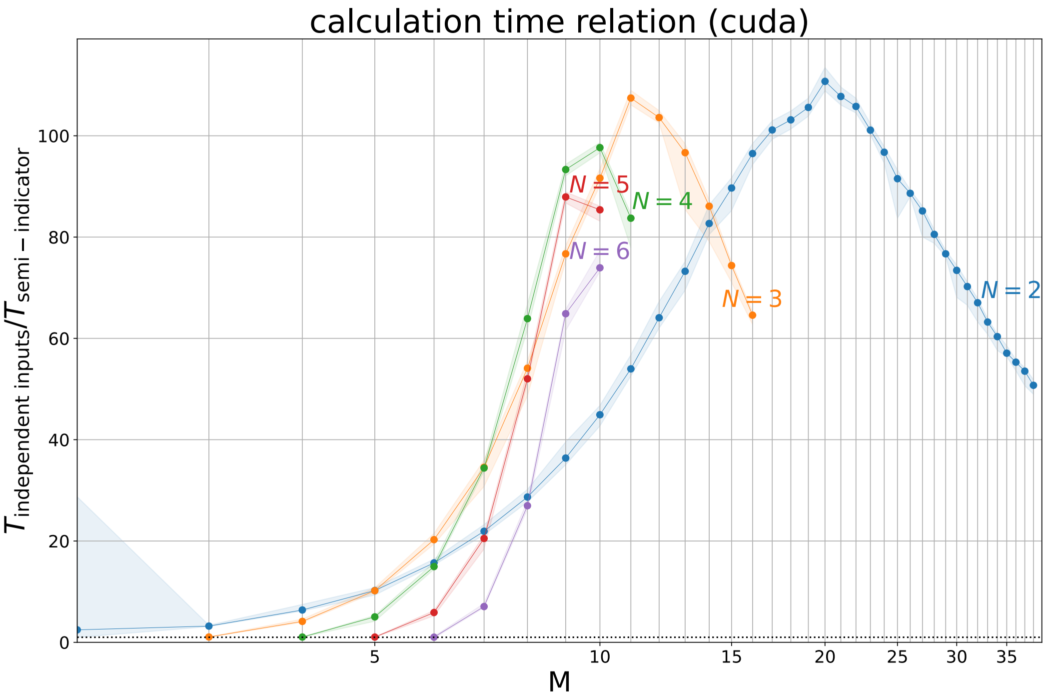


Both methods are asymptotically equivalent. For full bases: $O\left(N\binom{M-1+N}{N}^2\right)$.

Numerical comparison

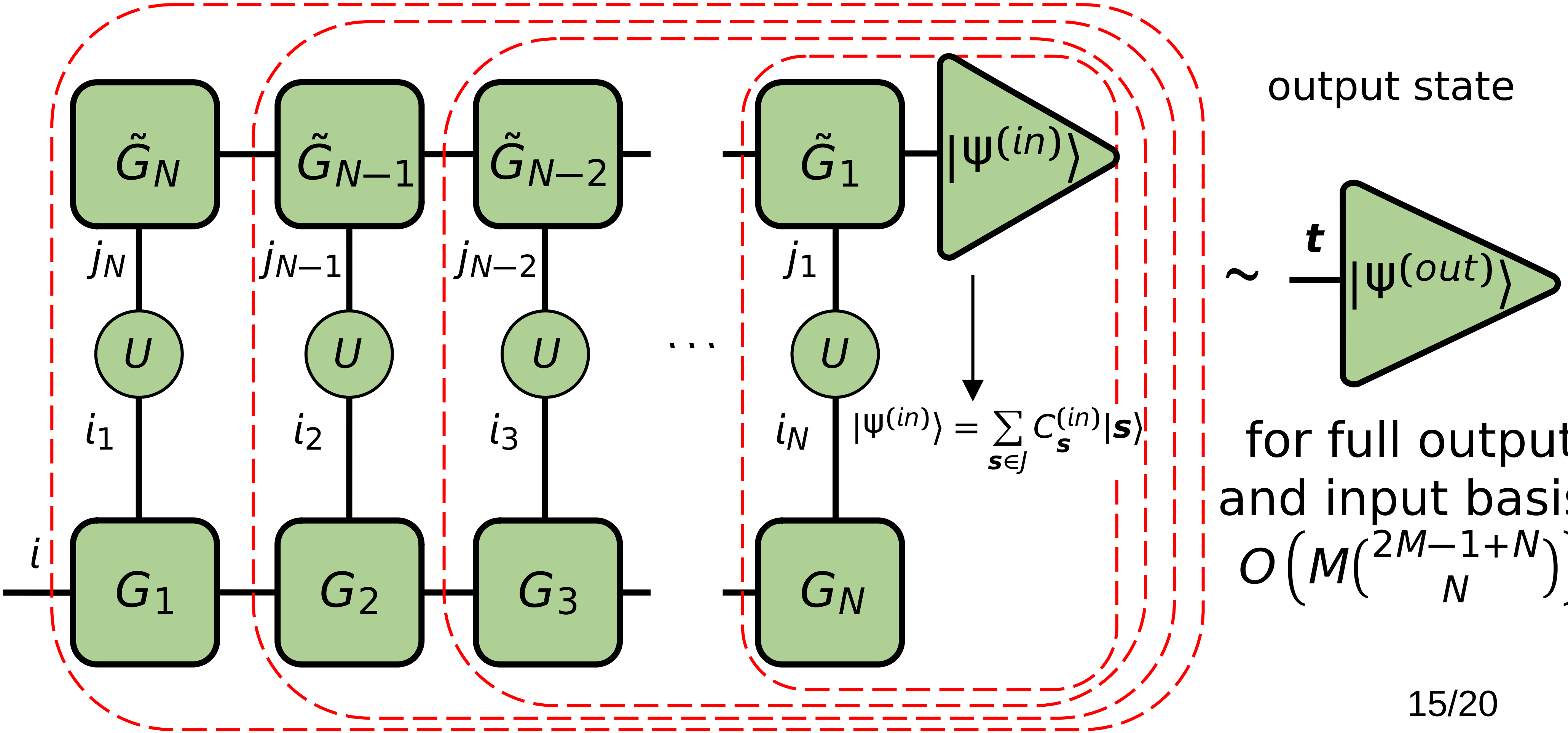
How faster is it to calculate transformation amplitudes with the semi-indicator tensor than to calculate each input separately for unbunched bases (assignment lists have no repetitions)?





Both methods are asymptotically equivalent.

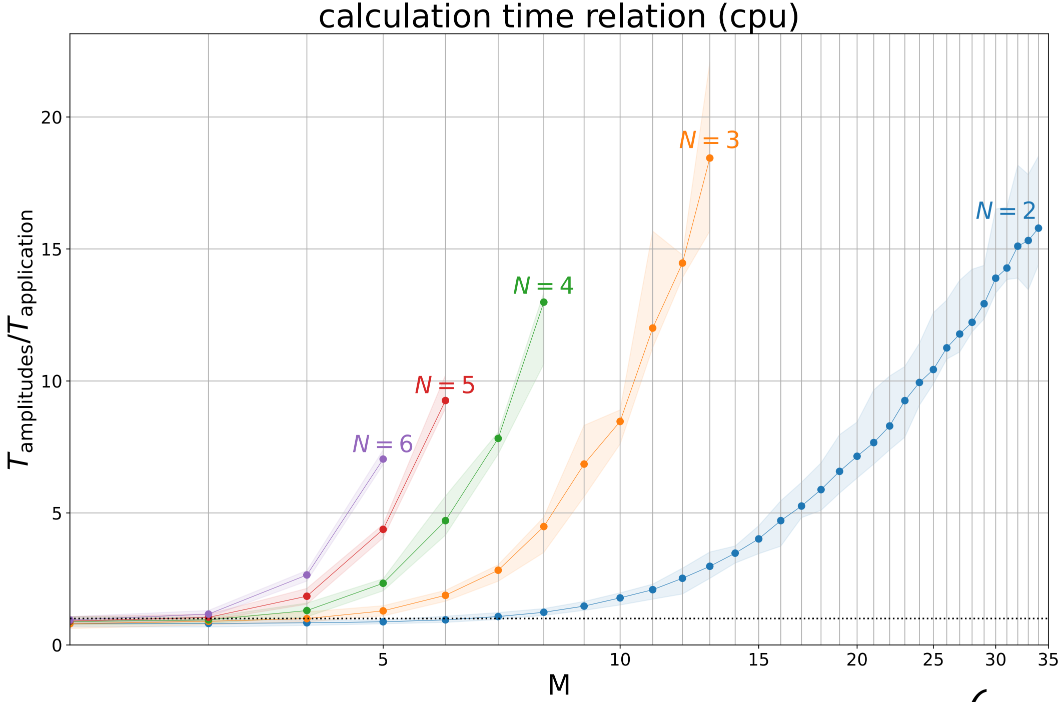
Arbitrary input, arbitrary output

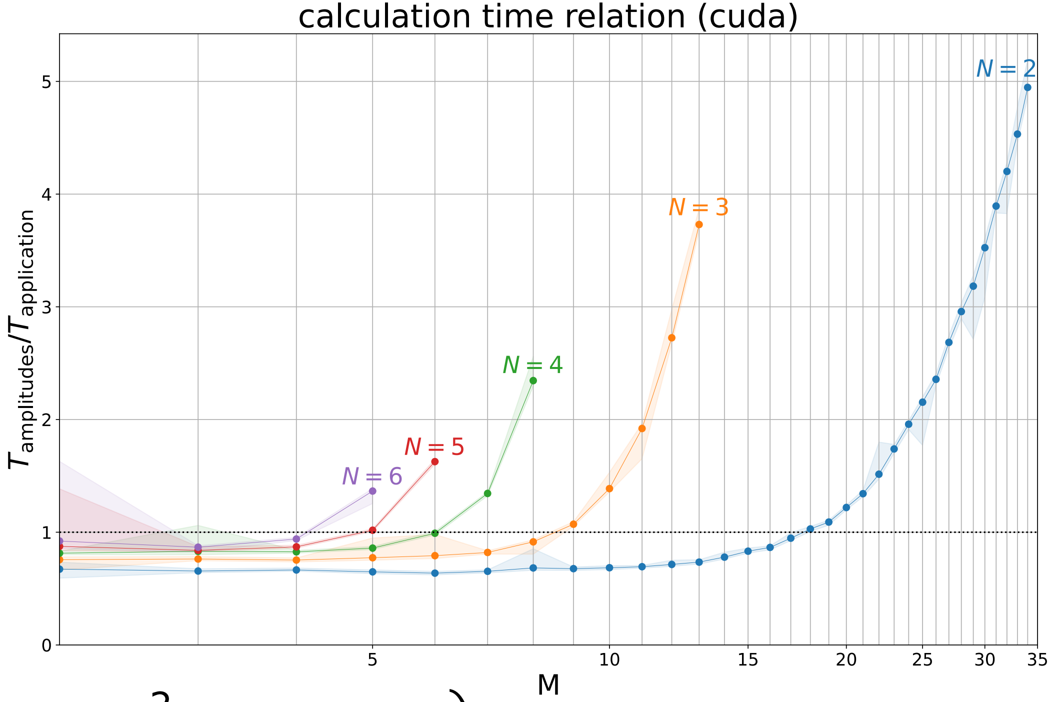


for full output and input basis: $O\left(M\binom{2M-1+N}{N}\right)$

Numerical comparison

How faster is it to calculate all the transformation amplitudes compared to calculating the application of the transformation for full bases?

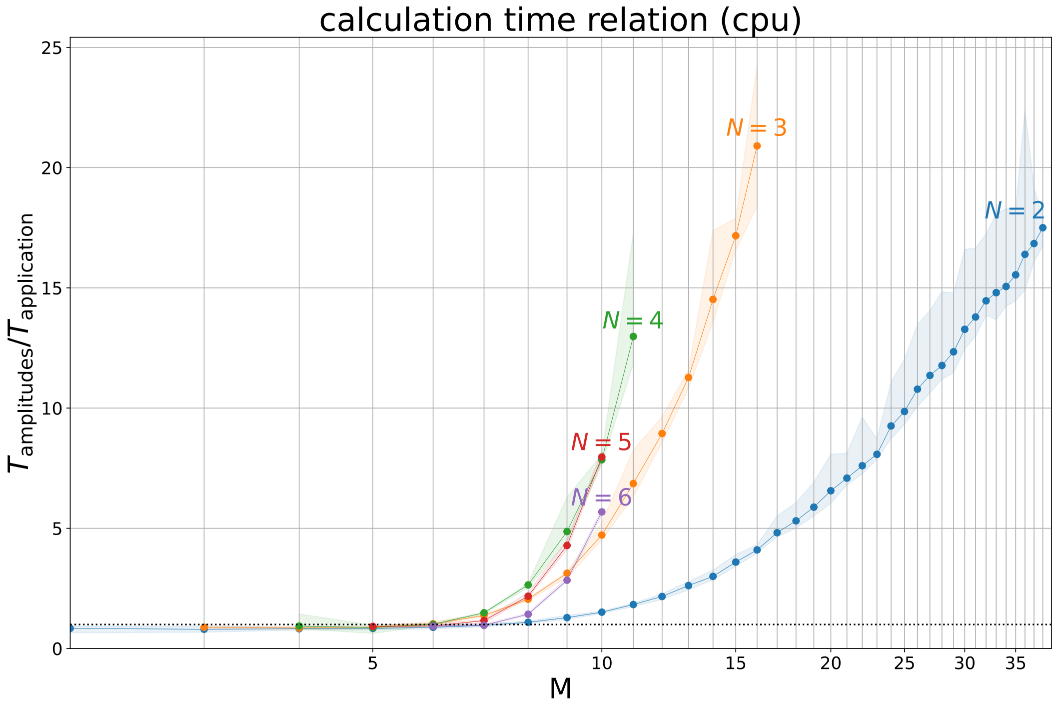


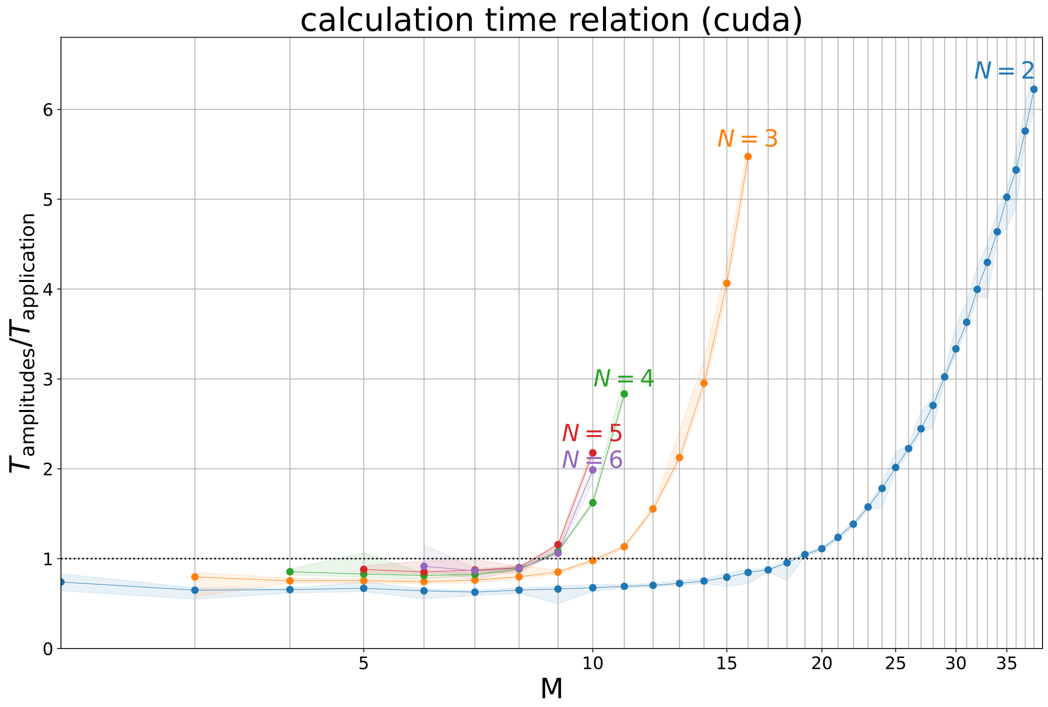


For full bases: $O\left(N\binom{M-1+N}{N}^2/M\binom{2M-1+N}{N}\right)$

Numerical comparison

How faster is it to calculate all the transformation amplitudes compared to calculating the application of the transformation for unbunched bases?





Possible generalization

Other quantities involving a sum over permutations

Distinguishable photons:

$$S_{ij} = \langle \phi_i | \phi_j \rangle$$

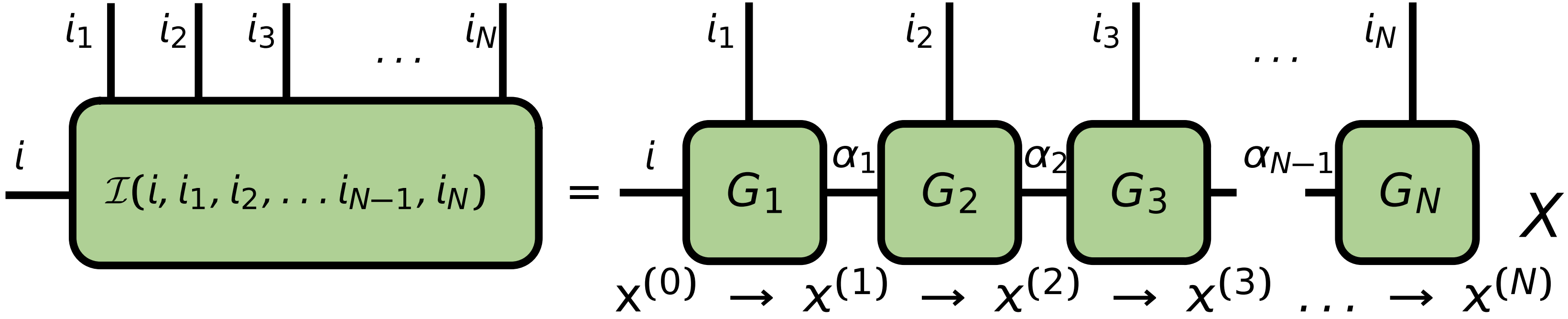
$$\mathcal{P}(\boldsymbol{t}|\boldsymbol{s}) = \sum_{\sigma,\tau \in \mathcal{S}_N} \prod_{i=1}^{N} S_{\sigma_i \tau_i} [\bar{U}^{\boldsymbol{ts}}]_{i\sigma_i} [U^{\boldsymbol{ts}}]_{i\tau_i}$$

Gaussian boson sampling:

$$\mathcal{P}(\boldsymbol{s}) = \frac{\mathrm{haf}[A^{\boldsymbol{s}}]}{\sqrt{\det[\sigma]}\boldsymbol{s}!}$$

$$\mathrm{haf}[A] = \frac{1}{M!2^M} \sum_{\sigma \in \mathcal{S}_{2M}} \prod_{i=1}^{M} A_{\sigma_{2i-1}\sigma_{2i}}$$

Derivative function decomposition



derivative function

$$X^{(k-1)} \xrightarrow{f_{i_k}^{(k)}} X^{(k)}$$

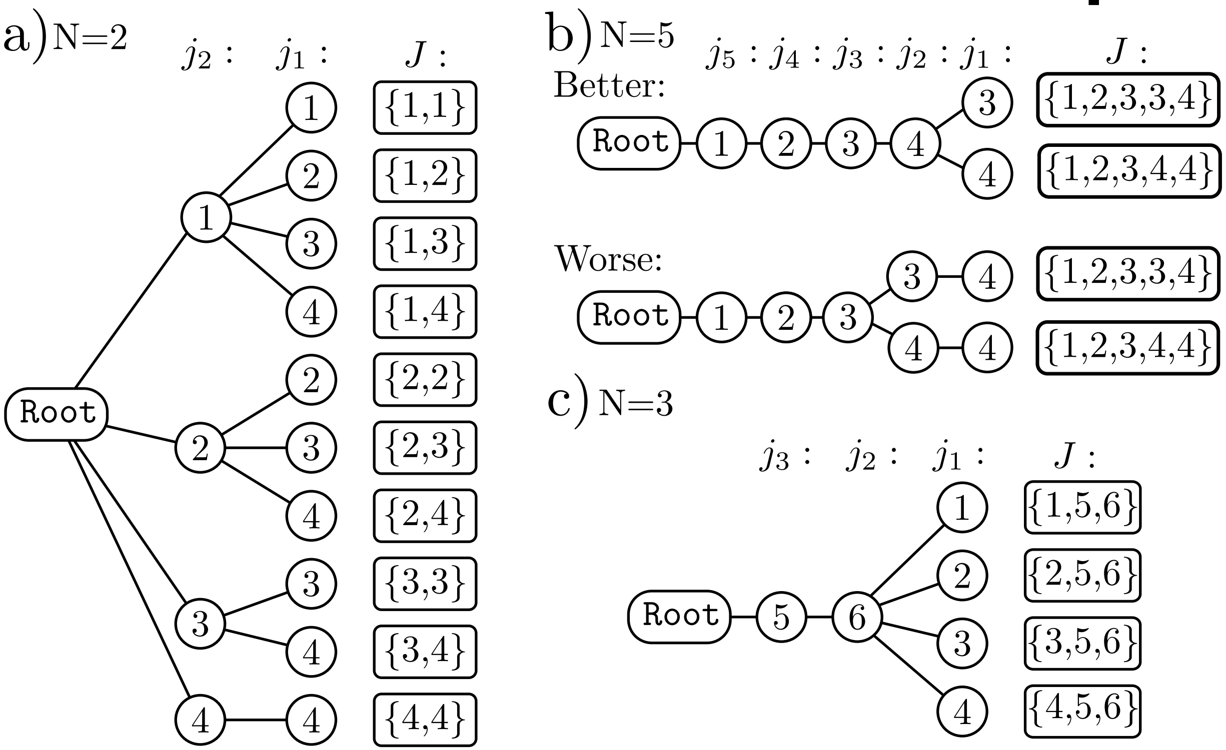
$$\mathrm{X}^{(0)} \sim L, x^{(0)} = \{T_1, T_2, T_3, \ldots T_M\} \sim L[i]$$

$$\alpha_{k-1} = \overline{1, |X^{(k-1)}|}, \alpha_k = \overline{1, |X^{(k)}|}$$

for $\mathrm{G}_k(\alpha_{k-1}; i_k; \alpha_k) = 0$ or $1; \forall 1 \leq k \leq N$:

$$x^{(k)} = f_{i_k}^{(k)}(x^{(k-1)}) = \begin{cases} x^{(k-1)}[i_k] \leftarrow x^{(k-1)}[i_k] - 1;\ x^{(k-1)}, & x[i_k] > 0, \\ \texttt{None}, & \text{else} \end{cases}$$

Derivative function decomposition



$$x^{(k)} = \tilde{f}_{j_k}^{(k)}(x^{(k-1)}) = \begin{cases} x^{(k-1)}.\text{parent}, & \text{if } x^{(k-1)}.\text{key} = j_k, \\ \text{None}, & \text{else} \end{cases}, \quad \forall 1 \le k \le N.$$